

**Note****Intersection Numbers for Coherent Configurations  
and the Spectrum of a Graph**

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It is shown that the intersection numbers of a coherent configuration are closely related to the determinant of a generic matrix for the corresponding adjacency algebra. This is important as both concepts provide isomorphism invariants for graphs.

The coherent configurations studied by Higman in [1] are of interest in graph theory because invariants for the isomorphism class of a graph are obtained in the intersection numbers of the unique minimal coherent configuration generated by the partition of  $V \times V$  induced by the edge set  $E$  and the diagonal set  $V$  of the vertices of the graph.

If  $A$  is the adjacency matrix of a labelled, directed graph  $\Gamma$  with vertex set  $V$ , one constructs the smallest algebra  $\mathcal{A}$  over  $\mathbb{C}$  containing  $A$ ,  $A^T$  and the characteristic matrices of sets  $S \subset V \times V$  which are maximal with respect to

$$(i, j) \text{ and } (k, l) \in S \text{ implies } a_{ij} = a_{kl} \text{ for all } (a_{rs}) \in \mathcal{A} \quad (1)$$

These sets  $S$  form the partition of  $V \times V$  which define a coherent configuration for  $\Gamma$  and their characteristic matrices (the 0, 1-adjacency matrices of the graphs with edge sets  $S$ ) generate  $\mathcal{A}$  as a module over  $\mathbb{C}$ . Let  $A_1, A_2, \dots, A_r$  be these matrices. Suppose  $X = \sum x_i A_i$  and  $Y = \sum y_j A_j$  are two generic matrices of  $\mathcal{A}$ . Then the intersection numbers are defined by

$$XY = \sum a_{ijk} x_i y_j A_k \quad (2)$$

They are clearly independent of the vertex numbering and so are isomorphism class invariants. The same is true of the characteristic polynomial  $\det(xI - A)$  and also of  $\det X$ , which certainly provides at least as much information as  $\det(xI - A)$ . The object of this paper is to show that the intersection numbers give a finer classification of graphs than does the

spectrum or indeed  $\det X$ , and to investigate how much weaker is the classification using  $\det X$ .

Suppose that  $A_i$  corresponds to  $S_i \subset V \times V$ . One may reasonably suppose that the numbers  $a_{ijk}$  are provided together with the partition of the sets  $S_i$  into diagonal and off-diagonal subsets. From (2) if  $S_d$  is a diagonal class with corresponding vertices  $V_d$ , then  $S_i \cap V_d \times V_d = S_i$  or  $\emptyset$  according as  $a_{idi} = 1 = a_{dii}$  or not. Let  $J_d = \sum' A_i$  where the sum is restricted to those  $i$  with  $S_i \subset V_d \times V_d$ . This is the matrix with 1's in the block  $V_d \times V_d$  and 0's elsewhere. Hence  $J_d^2 = |S_d| J_d$ . Equating coefficients of  $A_d$  using (2) yields the order of  $S_d$  as  $\sum' a_{ijd}$ , where  $i, j$  satisfy  $S_i, S_j \subset V_d \times V_d$ . Using (2) again one can inductively obtain the coefficients  $w_{dt}$  of each  $A_d$  for the powers  $X^t$  ( $t \leq |V|$ ). Now  $\text{trace}(X^t) = \sum_d w_{dt} |S_d|$  for the sum over the diagonal classes. Newton's formulae applied to the generic eigenvalues of  $X$  consequently yield  $\det X$ .

**THEOREM.** *The intersection numbers determine the determinant of a generic matrix in the adjacency algebra.*

Conversely, assume that  $\det X$  is given together with, as before, the partition into diagonal and off-diagonal sets. Let  $\mathcal{B}$  be the algebra over  $\mathbb{C}$  generated by  $B_i = A_i^T$  for  $1 \leq i \leq r$ . This is the algebra of the graph  $\Gamma^T$  with adjacency matrix  $B = A^T$ . Its intersection numbers satisfy  $b_{ijk} = a_{jik}$  and its generic matrix is  $Y = \sum x_i B_i = X^T$ . Because  $\det Y = \det X$ , the determinant cannot distinguish  $a_{ijk}$  from  $a_{jik}$ . However, it will be shown that  $a_{ijk} + a_{jik}$  is determined by  $\det X$ .

Let  $\{S_d | d \in D\}$  be the collection of diagonal sets with corresponding partition  $V = \bigcup_{d \in D} V_d$ . We will treat  $\det X$  as a polynomial in the variables  $x_d (d \in D)$ . The unique term  $z$  of highest degree has the form  $\prod_{d \in D} x_d^{r_d}$  and yields  $|S_d| = r_d$  for  $d \in D$ . For  $X = (x_{rs})$ , the coefficient of  $zx_i^{-1} x_j^{-1}$  for  $i, j \in D$  is  $-2^{-\delta} \sum_{r \in V_i} \sum_{s \in V_j} x_{rs} x_{sr} = -\frac{1}{2} \sum_k \delta_{ijk} |S_k| x_k x_{k'}$  where  $S_{k'} = S_k^T = \{(r, s) | (s, r) \in S_k\}$  defines  $k'$ ,  $\delta = 1$  or  $0$  according as  $i = j$  or not, and  $\delta_{ijk} = 1$  or  $0$  according as  $S_k \subset (V_i \times V_j) \cup (V_j \times V_i)$  or not. This pairs  $S_k$  with  $S_{k'}$ , establishes when  $S_k = S_{k'}$  and so yields  $|S_k|$  for all  $k$ .

The block of  $S_k$  is nearly determined: either  $V_i \times V_j$  or  $V_j \times V_i$  for  $i, j$  as above. Make an arbitrary choice of  $S_k \subset V_i \times V_j$  for one such triple  $(i, j, k)$  with  $i \neq j$ . This fixes the precise block of all other classes as follows: The action of  $X$  on  $\mathbb{C}^{|V|} = \mathbb{C}[v \in V]$  induces an action on the subspace  $\mathbb{C}^r$  ( $r = |D|$ ) with basis  $\{\sum_{v \in V_d} v | d \in D\}$  which is described by the matrix  $\bar{X}$  with entries  $\bar{x}_{de} = |V_e|^{-1} (\sum_{r \in V_d} \sum_{s \in V_e} x_{rs})$ . Because the variables in each block are distinct,  $\det \bar{X}$  is an irreducible factor of  $\det X$  with degree  $r$ . Set  $x_s = 0$  if  $S_s \cap V_d \times V_d = \emptyset$  for all  $d \in D$ , and  $x_s = 1$  if  $S_s \subset V_d \times V_d$  for some  $d \in D$  but  $s \notin D$ . Then  $\det X$  specialises to  $\prod_{d \in D} (x_d - 1)^{|V_d| - 1} (x_d + |V_d| - 1)$  and  $\det \bar{X}$  can be recognised as that factor which specialises

to  $\prod_{d \in D} (x_d + |V_d| - 1)$ . Now  $S_s \subset V_i \times V \cup V \times V_j$  if, and only if,  $\det \bar{X}$  does not contain a term with the product  $x_s x_k$  for  $i, j, k$  as chosen above. A couple of applications of this produces the block of every class  $S_s$  subject to the choice mentioned.

For the intersection numbers  $a_{uvw}$  where  $u$  or  $v \in D$  and  $w \notin D$  it is clear that  $a_{iww} = a_{wjw} = 1$  for  $S_w \subset V_i \times V_j$  and  $a_{uvw} = 0$  otherwise; and if  $k \in D$ , then  $a_{uu'k} = |S_u| |S_k|^{-1}$  for  $S_u \subset V_k \times V$  with  $a_{uvk} = 0$  otherwise. Any other intersection numbers contribute to the coefficient of  $z(x_i x_j x_k)^{-1}$  in  $\det X$  for some  $i, j, k \in D$ . This coefficient is obtained by looking at  $3 \times 3$  submatrices of  $X$  which include the three diagonal elements  $x_i, x_j, x_k$  and equals

$$(4-p)!^{-1} \sum_{S_u \subset V_i \times V_j} \sum_{S_v \subset V_j \times V_k} \sum_{S_w \subset V_k \times V_i} |S_w| (a_{uvw} x_u x_v x_w + a_{u'v'w} x_{u'} x_{v'} x_{w'}), \quad (3)$$

where  $p$  is the number of distinct indices among  $i, j, k$ ; the sums are restricted to off-diagonal classes (i.e.,  $u, v, w \notin D$ ); and  $S_{i'} = S_i^T$  defines  $u', v'$ , and  $w'$ . Symmetry in (3) for  $u, v, w$  ensures that

$$|S_u| a_{vwu'} = |S_v| a_{wuv'} = |S_w| a_{uvw'} \text{ and } a_{uvw} = a_{u'v'w'}, \quad (4)$$

For  $p > 1$  the blocks  $V_i \times V_j, V_j \times V_k$  and  $V_k \times V_i$  are distinct and so  $z(x_i x_j x_k)^{-1} x_u x_v x_w$  has coefficient  $|S_w| a_{uvw'}$ . This determines the intersection numbers except on diagonal blocks because those  $a_{uvw}$  not satisfying  $S_u \subset V_i \times V_j, S_v \subset V_j \times V_k$  and  $S_w \subset V_k \times V_i$  for some  $i, j, k \in D$  with  $u, v, w \notin D$  are necessarily zero.

Taking  $p = 1$  in (3) and letting  $q$  be the number of distinct indices among  $u, v, w$  yields the coefficient

$$(4-q)!^{-1} |S_w| (a_{uvw'} + a_{vuw'}) \quad (5)$$

for  $zx_i^{-3} x_u x_v x_w$  by virtue of (4). The intersection numbers for the diagonal blocks obtained via (5) are the only ones that cannot be individually ascertained once the choice of block for one off-diagonal  $S_s$  is made. If the alternative choice holds so that all the classes  $S_s$  have been associated with the transpose of their correct block, then  $a_{uvw}$  and  $a_{vuw}$  have to be interchanged throughout. However, in either case  $a_{uvw} + a_{vuw}$  is known.

**THEOREM.** *The determinant of a generic matrix in the adjacency algebra determines the sums  $a_{ijk} + a_{jik}$  of pairs of intersection numbers.*

Let  $\mathcal{A}_d$  be the algebra given by the block  $V_d \times V_d$ . Then  $\mathcal{A}_d$  is commutative if, and only if,  $a_{uvw} = a_{vuw}$  for all relevant  $u, v, w$ . In particular, this is the case when the rank or number of classes  $S_s \subset V_d \times V_d$  is sufficiently small. Whenever  $\mathcal{A}$  itself is commutative, the  $a_{uvw}$  can therefore

all be found. Thus the intersection numbers and  $\det X$  determine each other. For the important case when  $\Gamma$  is a regular (unlabelled) graph,  $\mathcal{A}$  is simply the algebra generated by the adjacency matrix  $A$  and the all 1's matrix  $J$ . So  $\mathcal{A}$  is commutative and  $\det X$  yields the same class of graphs as do the intersection numbers.

#### REFERENCE

1. D. G. HIGMAN, Coherent configurations, *Geom. Dedicata* **4** (1975), 1–32.